J. Fluid Mech. (1983), *vol.* 130, *pp.* 27–39
Printed in Great Britain

Weakly nonlinear interactions and wave trapping

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(Received 26 August 1981 and in revised form 30 November 1982)

When the flow over a submerged, round, upright cylinder, situated in a large ocean, is forced by a train of plane waves, linear theory (Yamamuro 1981) shows that the response can be abnormally large for certain forcing frequencies. The aim of this paper is to present a weakly nonlinear theory, where wave interactions, arising from the quadratic terms in the free-surface boundary conditions, can yield abnormally large responses.

A specific interaction will be considered between a flow at a subharmonic frequency and a flow at the driving frequency. The reason for considering such an interaction derived from a consideration of some experimental results of Barnard, Pritchard & Provis (1981).

1. Introduction

If a train of plane waves of a certain frequency is incident on a submerged upright cylinder, situated in a large ocean (figure 1), linear theory (Yamamuro–Renardy 1981) predicts that the overall amplitudes over the sill may become much larger than those of the deeper ocean. Such a phenomenon may be called 'wave trapping' or 'near-resonance', referring to the unusually large response. The theory of trapping of water waves in three-dimensional flows, where the bottom topographies model the natural variations of the ocean floors, is difficult (Meyer 1971, 1979). Hence the idealized problem of a submerged upright cylinder, or sill, has received the most analysis (Longuet-Higgins 1967; Summerfield 1969; Pite 1977; Renardy 1983). All these works are based on a linear theory. In order to verify the large near-resonances predicted by these theories. Barnard, Pritchard & Provis (1981) performed laboratory experiments and produced results which showed marked peaks at frequencies not associated with resonances predicted by linear theory. The purpose of this paper is to examine a contribution of nonlinear effects to the wave-trapping phenomenon.

When wave amplitudes are rather small, the strongest nonlinear effects are to be expected to arise from wave interactions deriving from the quadratic terms in the sill region (McEwan 1971; Mahony & Smith 1972). Thus it seems appropriate to restrict attention to these in an initial investigation as to whether nonlinear effects can contribute significantly to wave-trapping phenomena. Thus, if σ is the frequency of the incident waves, attention will be limited to interactions with waves of frequencies $\frac{1}{2}\sigma$ and 2σ. The velocity potential is then conveniently written as that of the linear solution plus the components excited by the nonlinear interactions. The calculation of the modal decomposition of the latter is complex and its difficulties are resolved in §§3 and 4. A possible subharmonic resonance is presented in §5 together with a particular laboratory condition under which it can occur.

The structure of the paper is as follows. In §2, the equations governing the flow are presented. Let the sill region ($r \leqslant 1$) be denoted by D_1 and the outer region ($r \geqslant 1$) by D_2. In order to calculate the amplitude of the wave motion, it is necessary to solve a 'homogeneous' problem in D_1, namely a linearized problem forced by an outer flow of a fixed frequency. This is investigated in §3. The problem posed in D_2 is solved from an existing linear theory. The flow structure in D_2 is then used to generate a boundary condition at the sill edge $r = 1$ so that the problem is reduced to a boundary-value problem in D_1. It is found from a separation of variables that an eigenstate in D_1, periodic in time with a complex-valued frequency Ω, consists of a 'wavelike' mode and an infinite number of 'decaying' modes which decay rapidly away from the sill edge. In addition, there are an infinite number of complex-valued coefficients to be determined from the boundary conditions. Two methods for calculating the frequencies Ω and the coefficients are presented. The first is a collocation method, which was found to be time-consuming. The second is a non-standard iterative scheme, based on the smallness of the response of the decaying modes compared with that of the wavelike modes.

In §4 the solution to the 'homogeneous' problem provides a set of orthogonal functions, a combination of which can be used to form an expression for the surface elevation. The total sill solution is then written as a suitable combination of the orthogonal functions, superposed on the linear solution.

In §5 an example of such a nonlinear interaction is constructed. The example chosen here displays subharmonic resonance and was motivated by previous work on edge waves (Guza & Davis 1974; Minzoni & Whitham 1977; Rockliff 1978). The interaction involves three modes: two $\cos 2\theta$ modes at the forcing frequency, and a $\cos\theta$ mode at half that frequency. It is found that near-resonance occurs for two ranges of forcing frequencies. One range occurs near an eigenfrequency of the $\cos 2\theta$ mode, and the other occurs near twice an eigenfrequency of the $\cos\theta$ mode.

2. Formulation of the problem

It is assumed that the motion is inviscid and irrotational. Let the origin be defined at the undisturbed water level above the centre of the submerged circular cylindrical sill. Let the z^* axis be taken upwards, the r^* axis outwards and let (r^*, θ, z^*) denote cylindrical coordinates. The radius of the sill is denoted by a, the depth of water above the sill is d, the depth outside the sill is D. Let the ratio d/D of the depths be δ. The dimensionless variables (without asterisks) are $\eta = \eta^*/d$, $z = z^*/d$, $r = r^*/a$, $\phi(r, \theta, z, t) = \phi^*(r^*, \theta, z^*, t)/d^2\sigma$, where $\eta(r, \theta, t)$ denotes the surface displacement and ϕ denotes the velocity potential. The geometry of the problem is shown in figure 1. The domain over the sill ($r \leqslant 1$) is denoted by D_1 and that outside the sill ($r \geqslant 1$) by D_2.

A train of plane waves of frequency σ and dimensionless amplitude $|\eta|_{\mathrm{I}}$ is incident on the sill from the positive x-axis ($\theta = 0$).

The velocity potential satisfies Laplace's equation:

$$\phi_{rr} + \frac{\phi_r}{r} + \frac{\phi_{\theta\theta}}{r^2} + \frac{a^2}{d^2}\phi_{zz} = 0. \tag{2.1}$$

The following boundary conditions apply. At large distances from the sill, the velocity potential must consist of the incident wave whose surface elevation is the real part of

$$\eta = |\eta|_{\mathrm{I}}\, \mathrm{e}^{-\mathrm{i}(kx+\sigma t)}, \tag{2.2}$$

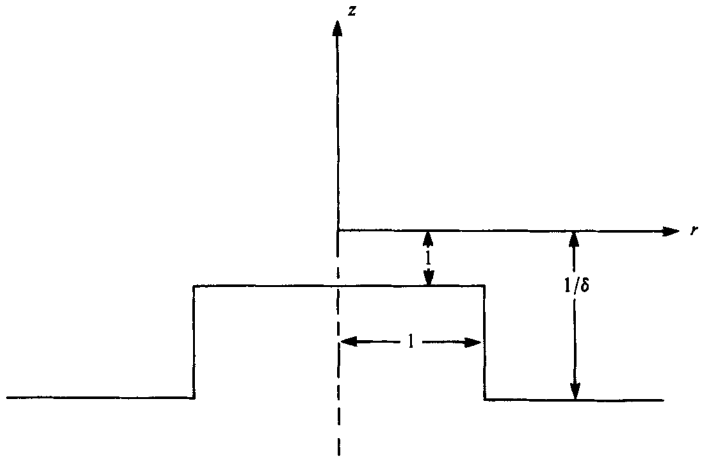

FIGURE 1

where k is the positive real root of

$$\frac{kD}{a}\tanh\frac{kD}{a} = \frac{D\sigma^2}{g}, \tag{2.3}$$

and waves which travel or decay outwards.

The free-surface boundary conditions expanded about the equilibrium level and retaining quadratic terms are (Phillips 1966), for $z = 0$, $r \geqslant 0$, $0 \leqslant \theta \leqslant 2\pi$, the kinematic condition

$$\eta_t = \sigma\phi_z + \sigma\eta\phi_{zz} - \frac{d^2\sigma}{a^2}\left(\phi_r, \frac{\phi_\theta}{r}\right)\cdot\left(\eta_r, \frac{\eta_\theta}{r}\right), \tag{2.4}$$

and

$$\phi_{tt} + \frac{g}{d}\phi_z = -\eta\frac{\mathrm{d}}{\mathrm{d}z}\left(\phi_{tt} + \frac{g}{d}\phi_z\right) - \frac{1}{d^2\sigma}\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{u}^*\cdot\boldsymbol{u}^*), \tag{2.5}$$

where

$$\boldsymbol{u}^* = d^2\sigma\left(\frac{\phi_r}{a}, \frac{\phi_\theta}{ar}, \frac{\phi_z}{d}\right). \tag{2.6}$$

In order to calculate the final solution ϕ, it is convenient to find an expression for η before conditions (2.5) is used. The following ordering argument shows that condition (2.4) can be linearized. In (2.4), η_t is essentially $-\mathrm{i}\omega\eta$ if ω is the frequency of the flow. Thus (2.4) yields η in the form $-\mathrm{i}\omega\eta = d^2\sigma\phi_z +$ terms quadratic in ϕ. When this is substituted into (2.5) for η, it can be seen that the quadratic terms of ϕ in η contribute cubic terms to (2.5), which will be neglected. Hence the only terms in (2.4) that affect the final solution are the linear terms, so that $\eta_t = \sigma\phi_z$ for the calculation of ϕ.

Outside the sill, the linearized conditions will be applied since we consider geometries in which the ratio of the sill depth to the outer depth is small, and the wave amplitudes outside the sill are assumed to be rather small. Hence we restrict attention to weakly nonlinear effects solely in the sill region. Under these assumptions, the free-surface boundary conditions are, for $z = 0$, $r \geqslant 0$, $0 \leqslant \theta \leqslant 2\pi$,

$$\phi_{tt} + \frac{g}{d}\,\phi_z = \begin{cases} -\eta\dfrac{\mathrm{d}}{\mathrm{d}z}\left(\phi_{tt} + \dfrac{g}{d}\phi_z\right) - \dfrac{1}{d^2\sigma}\dfrac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{u}^*\cdot\boldsymbol{u}^*) \quad \text{in } D_1, \\ 0 \quad \text{in } D_2, \end{cases} \tag{2.7}$$

and

$$\eta_t = \sigma\phi_z(1+O(\text{slope})) \quad \text{in } D_1 \text{ and } D_2. \tag{2.8}$$

Equation (2.8) can be used to recover the surface displacement if it is needed. As mentioned in §1, only flows of frequencies σ, 2σ and $\frac{1}{2}\sigma$ will be investigated, so that condition (2.7) can be expressed in the following form:

$$\phi_{tt}+\frac{g}{d}\phi_z = \left\{\begin{array}{ll} f_1(r,\theta)\,\mathrm{e}^{-\mathrm{i}\sigma t}+f_2(r,\theta)\,\mathrm{e}^{-\frac{1}{2}\mathrm{i}\sigma t}+f_3(r,\theta)\,\mathrm{e}^{-\mathrm{i}2\sigma t}+* & \text{for } D_1 \\ 0 \quad \text{for } D_2. & \end{array}\right\} \tag{2.9}$$

Here, and in what follows, the asterisk denotès the complex conjugate of preceding terms. f_1, f_2 and f_3 are complex-valued functions of ϕ. They are thus determined once an expression for ϕ is known and then substituted into (2.7) and (2.8), retaining only quadratic terms.

The boundary conditions on rigid surfaces are:

at $z = 1$, $0 \leqslant r < 1$ and $z = -1/\delta$, $r > 1$,

$$\phi_z = 0; \tag{2.10}$$

at $r = 1$, $-1/\delta < z < -1$,

$$\phi_r = 0. \tag{2.11}$$

At $r = 1$, $-1 < z < 0$, $0 \leqslant \theta \leqslant 2\pi$, the velocity is continuous, a condition which is equivalent to the continuity of ϕ and ϕ_r there.

Consider the response of a solution of frequency ω ($\omega = \sigma$, $\frac{1}{2}\sigma$ or 2σ). The linear terms in condition (2.9) show that the principal response of that mode is inversely proportional to $-\omega^2+\Omega^2$, where Ω is the associated eigenfrequency. Thus resonance is possible if $-\omega^2+\Omega^2$ is small and begins to become comparable to the dimensionless amplitude. Hence the homogeneous problem must be solved for the complex eigenfrequencies, of which there are a countable number (Meyer 1971, §4).

The standard analysis for evaluating the eigenfrequencies Ω involves expressing the (r, z)-dependence of the modes which vary as $\cos m\theta\, \mathrm{e}^{-\mathrm{i}\Omega t}$ as a linear combination of an infinite number of functions separable in r and z. These eigenfunctions are to satisfy the linearized conditions and are not forced by the plane waves. The conditions at $r = 1$ then yield an infinite matrix equation, whose zeros must be found in order to solve for Ω. These eigenfunctions generalize the 'free modes' investigated by Longuet-Higgins (1967) using shallow-water theory. However, it is found that these eigenfunctions are not useful for the construction of the solution because they grow nearly exponentially with r and are not orthogonal over any finite range of r. Hence, when these are substituted into (2.9), an integration over r will not yield the response of that mode. It is shown in §§3 and 4 that these difficulties are avoided if attention is confined to the range $r \leqslant 1$. If ϕ_{s} denotes the solution to the linearized problem in the sill region, the flow stucture outside the sill can be used to generate a boundary condition at $r = 1$, giving ϕ_{s} in terms of $\mathrm{d}\phi_{\mathrm{s}}/\mathrm{d}r$. The admissible values for the complex frequencies in ϕ_{s} are then found from this boundary condition, and the modes of ϕ_{s} can be used in the construction of the solution for the full problem.

3. The 'homogeneous' problem

As indicated in §1, the discussion is focused on flows at the frequencies σ, $\frac{1}{2}\sigma$ and 2σ, σ being the driving frequency. In the following, ω denotes one of the three frequencies.

The problem formulated in §2 is essentially a superposition of two problems. Hence the solutions are expressed as $\phi = \phi_{\mathrm{L}}+\phi_{\mathrm{N}}$. ϕ_{L} satisfies the linearized free-surface conditions, namely at $z = 0$, $0 \leqslant \theta \leqslant 2\pi$, $0 \leqslant r$

$$\phi_{\mathrm{L}tt} + \frac{g}{d}\phi_{\mathrm{L}z} = 0, \tag{3.1}$$

and is forced by the plane waves represented by (2.2) and (2.3). ϕ_{L} has been calculated (Yamamuro–Renardy 1981) by a separation of variables for the regions D_1 and D_2, and the velocity is made continuous throughout the flow. ϕ_{N} is constructed in §4 to make the total velocity potential ϕ satisfy the free-surface conditions (2.7) and (2.8). Both ϕ_{L} and ϕ_{N} satisfy conditions (2.10) and (2.11) at the solid walls. ϕ_{N} and ϕ_{L} interact through the free-surface conditions, and the solutions of interest are those in which ϕ_{N} becomes comparable to ϕ_{L}.

In order to construct ϕ_{N}, it is convenient to first consider a 'homogeneous' problem in D_1. This problem of a linearized flow, forced at $r = 1$ by a boundary condition that reflects a flow in D_2 of a fixed frequency ω, will now be described together with the behaviour of its solutions. The actual construction of ϕ_{N} in D_1 from these solutions is presented in §4. Before posing the 'homogeneous' problem, the boundary condition at $r = 1$ for D_1 with an outer flow of a given frequency ω (taken to denote σ, $\frac{1}{2}\sigma$ or 2σ) will be presented. Let ϕ_1 denote the flow in D_1 and ϕ_2 the flow in D_2. Since the flow is linear, the outer flow has the form (Yamamuro–Renardy 1981)

$$\phi_2 = \mathrm{e}^{-\mathrm{i}\omega t} \sum_{m=0}^{\infty} \cos m\theta \Big[B_{m0} H_m^{(1)}\Big(\frac{a\lambda r}{D}\Big) \cosh \lambda(\delta z+1) + \sum_{n=1}^{\infty} B_{mn} \cos \lambda_n(\delta z+1)\, K_m\Big(\frac{a\lambda_n r}{D}\Big)\Big/ K_m\Big(\frac{a\lambda_n}{D}\Big)\Big] + *, \tag{3.2}$$

where λ and λ_n are the roots of the dispersion relation (Davis & Hood 1976)

$$\begin{Bmatrix} \lambda \\ \mathrm{i}\lambda_n \end{Bmatrix} \tanh \begin{Bmatrix} \lambda \\ \mathrm{i}\lambda_n \end{Bmatrix} = \frac{D\omega^2}{g}. \tag{3.3}$$

The coefficients B_{mn} are as yet undetermined. The notations for the Bessel functions are those used by Abramowitz & Stegun (1972). The continuity of velocity at $r = 1$, $-1 < z < 0$ yields

$$\frac{\mathrm{d}\phi_2}{\mathrm{d}r} = \begin{cases} \dfrac{\mathrm{d}\phi_1}{\mathrm{d}r} & (-1 < z < 0), \\ 0 & (-1/\delta < z < 1), \end{cases} \tag{3.4}$$

$$\phi_2 = \phi_1 \quad (-1 < z < 0). \tag{3.5}$$

The expression (3.2) is substituted in (3.4). The orthogonality of the set $\{\cosh \lambda(\delta z+1), \cos \lambda_n(\delta z+1); n = 1, 2, \ldots\}$ over $-1/\delta < z < 0$ is used, and integration over z yields the B_{mn} in terms of $\mathrm{d}\phi_1/\mathrm{d}r$ at $r = 1$. These equations are then used in (3.5) to eliminate the B_{mn}. A boundary condition for $\phi_m(r,z)$, the coefficient of $\cos m\theta\, \mathrm{e}^{-\mathrm{i}\omega t}$ in ϕ_1, is obtained: at $r = 1$, $-1 < z < 0$,

$$\phi_m(r,z) = \int_{-1}^{0} \left[\frac{\mathrm{d}\phi_m(r,z')}{\mathrm{d}r}\right]_{r=1} K_\omega(z,z')\, \mathrm{d}z', \tag{3.6}$$

where

$$K_\omega(z,z') = \frac{H_m(a\lambda/D) \cosh \lambda(\delta z+1) \cosh \lambda(\delta z'+1)}{H_m'(a\lambda/D)\,(a\lambda/D)\, h(\lambda)} + \sum_{n=1}^{\infty} \frac{K_m(a\lambda_n/D) \cos \lambda_n(\delta z+1) \cos \lambda_n(\delta z'+1)}{K_m'(a\lambda_n/D)\,(a\lambda_n/D)\, h(\mathrm{i}\lambda_n))}. \tag{3.7}$$

The 'homogeneous' problem in D_1, for each of the frequencies ω, consists of the boundary conditions (3.6) and

$$\phi_{tt} + \frac{g}{d}\phi_z = 0 \quad (z = 0),$$

$$\phi_z = 0 \quad (z = -1).$$

Separation of variables yields solutions of the form

$$\Phi_{mn}(r, z) \cos m\theta \, e^{-i\Omega_{mn} t},$$

where

$$\Phi_{mn}(r, z) = A_m J_m\left(\frac{akr}{d}\right) \cosh k(z+1) + \sum_{p=1}^{\infty} A_{mp} I_m\left(\frac{ak_p r}{d}\right) \cos k_p(z+1) \Big/ I_m\left(\frac{ak_p}{d}\right), \tag{3.8}$$

with

$$\begin{Bmatrix} k \\ ik_p \end{Bmatrix} \tanh \begin{Bmatrix} k \\ ik_p \end{Bmatrix} = \frac{d\Omega_{mn}^2}{g}. \tag{3.9}$$

Here Ω_{mn} denotes the nth largest frequency which is determined by condition (3.6). The ratios of the coefficients A_{mp}/A_m are also determined by (3.6), and A_m is left undetermined. Two methods for their computation will now be described, together with the orthogonality of the set $\{\Phi_{mn}(r,0);\, m \text{ fixed},\, n = 1, 2, \ldots\}$ for $0 \leqslant r \leqslant 1$.

Substitution of (3.8) into condition (3.6) yields the following equations for the unknowns A_m, A_{mn}, k and k_n:

$$A_m L(k, z) + \sum_{n=1}^{\infty} A_{mn} L_n(k_n, z) = 0 \quad (-1 < z < 0), \tag{3.10}$$

where the operators $L(k, z)$ and $L_n(k_n, z)$ are defined in the appendix. The wavenumbers λ and λ_n appearing in L and L_n are obtained through (3.3) for any one value of ω (σ, $\frac{1}{2}\sigma$ or 2σ). A matrix equation for A_m and A_{mn} is constructed by satisfying (3.10) at N values of z and neglecting the coefficients A_{mn} for $n \geqslant N$. This yields an $N \times N$ matrix. The Ω's can be obtained by a search through the complex plane as follows. The condition number of the matrix is computed over a grid of complex frequencies and those with relatively large condition numbers are taken to be approximations to the Ωs. The matrix equation for A_{mn}/A_m can then be solved. Convergence with N was checked numerically. Since this procedure is time-consuming, an alternative method based on an interative scheme was devised.

The iterative scheme is constructed to take advantage of the largeness of the response of the wavelike modes, represented by the Bessel functions J_m, as compared with the response $|A_{mp}|$ of the decaying modes, and of the property that $|A_{mp}|$ is smaller for larger p; i.e. the wavefield contains little of the modes that decay very fast away from the sill-edge. Equation (3.10) is multiplied by each of the functions of the set $\{\cosh k(z+1), \cos k_n(z+1);\, n = 1, 2, \ldots\}$ and integrated over $-1 \leqslant z \leqslant 0$. Since the elements of that set are orthogonal to each other, the resulting equations take the form

$$A_m X(k, \lambda, \lambda_n) + \sum_{s=1}^{\infty} A_{ms} X_s(k_s, \lambda, \lambda_n) = 0 \quad (m = 0, 1, 2, \ldots), \tag{3.11}$$

$$A_{mp} Y(k_p, k_p, \lambda, \lambda_n) + \sum_{\substack{s=1 \\ s \neq p}}^{\infty} A_{ms} Y(k_s, k_p, \lambda, \lambda_n) = A_m Y(k, k_p, \lambda, \lambda_n) \quad (p = 1, 2, \ldots). \tag{3.12}$$

The functions X, X_p and Y are defined in the appendix.

The iteration proceeds as follows. The first iterate for k, denoted by $k^{(0)}$, is calculated from the following reduced form of (3.11), in which all decaying modes are neglected:

$$J_m\left(\frac{ak}{d}\right)H'_m\left(\frac{a\lambda}{D}\right)f(k)\,\delta h(\lambda)\,\lambda-kJ'_m\left(\frac{ak}{d}\right)H_m\left(\frac{a\lambda}{D}\right)[g(\lambda,k)]^2=0, \tag{3.13}$$

where λ is known. The corresponding frequency $\Omega^{(0)}$ is calculated from

$$k^{(0)}\tanh k^{(0)}=\frac{d\Omega^{(0)2}}{g}, \tag{3.14}$$

after which the $k_n^{(0)}$ are calculated from (3.9). Next, the equations (3.12) are used to express A_{mn}/A_m for $n = 1, 2, \ldots$. Then, on using these relations to eliminate the A_{mn}, (3.11) takes the form:

$$x_1(k)+\delta\lambda h(\lambda)\,x_2(k,k_n)\,H'_m\left(\frac{a\lambda}{D}\right)=0, \tag{3.15}$$

where $x_1(k)$ represents the left-hand side of (3.13) and contains no decaying modes, $x_2(k, k_n)$ involves the decaying modes, and the notation is defined in the appendix.

The nth iterate $k^{(n)}$ $(n = 1, 2, \ldots)$ is calculated by Newton's method from (3.15) in which the term $x_2(k, k_n)$ is calculated at the known $(n-1)$th iterate, i.e.

$$x_1(k^{(n)})+\delta\lambda h(\lambda)\,x_2(k^{(n-1)},k_m^{(n-1)})\,H'_m\left(\frac{a\lambda}{D}\right)=0. \tag{3.16}$$

The corresponding $\Omega^{(n)}$ is then calculated from (3.14) with superscript n instead of 0, after which (3.9) yields the $k_m^{(n)}$.

A numerical check of the scheme was performed as follows. The Ωs were compared with the eigenfrequencies for the entire domain, described towards the end of §2 because they were expected to have similar values. The eigenfrequencies were calculated in a similar way to the Ωs. The only difference was that, in (3.13) and (3.16), λ was also an unknown. Therefore, at each step of the iteration, the two equations (3.16) and

$$k\tanh k-\delta\lambda\tanh\lambda=0 \tag{3.17}$$

where solved simultaneously for k and λ by Newton's method.

It is next shown that the set $\{\Phi_{mn}(r,0);\ m \text{ fixed},\ n = 1, 2, \ldots\}$ defined in (3.8) is orthogonal for $0 \leqslant r \leqslant 1$. Let $\Phi_{mp}(r,z)\cos m\theta\, e^{-i\Omega_{mp}t}$ and $\Phi_{mq}(r,z)\cos m\theta\, e^{-i\Omega_{mq}t}$ be distinct $(\Omega_{mp} \neq \Omega_{mq})$ velocity potentials satisfying the 'homogeneous' problem. An application of Green's theorem to the volume of fluid D_1 yields

$$0=\left(\frac{d\Omega^2_{mp}}{g}-\frac{d\Omega^2_{mq}}{g}\right)\int_{r=0}^{1}\int_{\theta=0}^{2\pi}[\Phi_{mp}\Phi_{mq}]_{z=0}\,r\,dr\,d\theta$$
$$+\frac{d^2}{a^2}\int_{z=-1}^{0}\int_{\theta=0}^{2\pi}\left[\Phi_{mq}\left(\frac{d\Phi_{mp}}{dr}\right)-\Phi_{mp}\left(\frac{d\Phi_{mq}}{dr}\right)\right]_{r=1}dz\,d\theta. \tag{3.18}$$

In the integrals at $r = 1$, $\Phi_{mq}(1,z)$ and $\Phi_{mp}(1,z)$ are replaced by expressions involving $d\Phi_{mq}(1,z)/dr$ and $d\Phi_{mp}(1,z)/dr$ with the use of (3.6). Further, the property that $K_\omega(z,z') = K_\omega(z',z)$ in (3.7) is used to reduce (3.18) to

$$\int_0^1[\Phi_{mp}\Phi_{mq}]_{z=0}\,r\,dr=0.$$

Therefore the set $\{\Phi_{mn}(r,0);\ m \text{ fixed},\ n = 1, 2, \ldots\}$ is orthogonal for $0 \leqslant r \leqslant 1$. These functions will be used to represent ϕ_N.

4. Form of 'resonant' solutions

The ϕ_N introduced in §3 must satisfy conditions (2.7), (2.8), (2.10), (2.11) and the radiation condition that the flow be non-growing at large r. Let the part of ϕ_N in D_1 be denoted by ϕ_{N1} and that in D_2 by ϕ_{N2}. ϕ_{N2} then satisfies linearized conditions. Both ϕ_{N1} and ϕ_{N2} consist of flows of frequencies σ, $\frac{1}{2}\sigma$ and 2σ. Hence ϕ_{N2} is a superposition of the expressions (3.2) for $\omega = \sigma, \frac{1}{2}\sigma$ and 2σ. Attention will be focused on the sill region where

$$\phi_{N1} = e^{-\frac{1}{2}i\sigma t}\,\Psi_1(r,\theta,z) + e^{-i\sigma t}\,\Psi_2(r,\theta,z) + e^{-i2\sigma t}\,\Psi_3(r,\theta,z) + *, \tag{4.1}$$

and Ψ_1, Ψ_2 and Ψ_3 must be determined. The continuity of ϕ_N at $r = 1$ yields the boundary condition (3.6) for ϕ_{N1}, namely the $\cos m\theta$ modes in Ψ_1, Ψ_2 and Ψ_3 must satisfy (3.6) for $\omega = \frac{1}{2}\sigma$, σ and 2σ respectively. At $z = 0$, $\phi_{N1} + \phi_L$ satisfies (2.7) and (2.8). At $z = -1$, $d\phi_{N1}/dz = 0$. Hence

$$\phi_{N1} = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}[\alpha_{mn}\,e^{-\frac{1}{2}i\sigma t}\,\Phi_{\alpha mn}(r,z) + \beta_{mn}\,e^{-i\sigma t}\,\Phi_{\beta mn}(r,z) + \gamma_{mn}\,e^{-i2\sigma t}\,\Phi_{\gamma mn}(r,z)]\cos m\theta + *, \tag{4.2}$$

where the sets $\{\Phi_{\alpha mn}(r,z)\}$, $\{\Phi_{\beta mn}(r,z)\}$ and $\{\Phi_{\gamma mn}(r,z)\}$ represent the solutions (3.8) of the 'homogeneous' problem for $\omega = \frac{1}{2}\sigma$, σ and 2σ respectively, with α_{mn}, β_{mn}, γ_{mn} being the response coefficients of their wavelike modes. In addition, the orthogonality of these sets of functions is used to calculate the coefficients α_{mn}, β_{mn} and γ_{mn} from the conditions at $z = 0$. For example, using the notation of (2.9),

$$[-\tfrac{1}{4}\sigma^2 + \Omega^2_{\alpha mn}]\,\alpha_{mn}\int_0^1 \Phi^2_{\alpha mn}(r,0)\,r\,dr = \int_0^1 \Phi_{\alpha mn}(r,0)\left[\frac{\int_0^{2\pi} f_2(r,\theta)\cos m\theta\,d\theta}{\int_0^{2\pi}\cos^2 m\theta\,d\theta}\right] r\,dr, \tag{4.3}$$

where $\Omega_{\alpha mn}$ is defined in (3.9) and corresponds to the solution $\Phi_{\alpha mn}$ given by (3.8) with $\omega = \frac{1}{2}\sigma$. Similar equations yield β_{mn} and γ_{mn}.

In the representation of ϕ_{N1} in (4.2), only a few of the constants α_{mn}, β_{mn}, γ_{mn}, need to be determined: the others are negligible. For example, it can be seen from (4.3) that the response $|\alpha_{mn}|$ is inversely proportional to $|-\frac{1}{4}\sigma^2 + \Omega^2_{\alpha mn}|$, so that, in computations involving a particular geometry, it may be sufficient to include only the one α_{mn} whose $\Omega_{\alpha mn}$ is the closest to $\frac{1}{2}\sigma$. Similarly, only the β_{mn} for which $\Omega_{\beta mn}$ is closest to σ and the γ_{mn} for which $\Omega_{\gamma mn}$ is closest to 2σ are expected to be significant enough for inclusion in the interaction calculations. This simplifies ϕ_{N1}. An example of the simplest subharmonic resonance is an interaction of the $\cos 2\theta\, e^{-i\sigma t}$ mode and the $\cos\theta\, e^{-\frac{1}{2}i\sigma t}$ mode and is described in §5.

5. Example of a near-resonance

A particular set of conditions in which the foregoing theory yields near-resonance will be presented. In order to simplify computations, the parameters δ and d/a will be chosen to be small. The smallness of d/a ensures the smallness of the effect of the decaying modes in the flow in D_1, so that in the 'interaction' equations, such as (4.3), the decaying modes will be assumed to be negligible. However, the decaying modes will not be neglected in the computation of the Ωs since these are required to a high order of accuracy. Furthermore, the smallness of δ ensures that some of the Ωs will have very small imaginary parts, so that if the flow is forced near such an eigenfrequency, near-resonance is possible.

The experimental scales of Barnard *et al.* (1981) were examined for the presence of near-resonant nonlinear interactions but were found not to yield them. However, a choice of scales that do is

$$d = 2\ \mathrm{cm}, \quad \frac{d}{a} = 0.005, \quad \delta = 0.002, \quad |\eta|_{\mathrm{I}} = 0.01. \tag{5.1}$$

In this case, the maximum amplitudes of the decaying modes is at least an order of magnitude less than that of the wavelike mode. Although these scales are unusual, there may be other realistic combinations where the nonlinear theory is applicable.

An interaction of three modes will be considered: the $(\cos\theta\,\mathrm{e}^{-\frac{1}{2}\mathrm{i}\sigma t}+*)$ and $(\cos 2\theta\,\mathrm{e}^{-\mathrm{i}\sigma t}+\,*)$ modes in ϕ_{N} and the $(\cos 2\theta\,\mathrm{e}^{-\mathrm{i}\sigma t}+*)$ mode in ϕ_{L}. In the sill region these are represented as follows:

$$\phi_{\mathrm{L}} = A_2 J_2\left(\frac{akr}{d}\right)\cos 2\theta\cosh k(z+1)\,\mathrm{e}^{-\mathrm{i}\sigma t}+*+\text{decaying modes}, \tag{5.2}$$

$$\phi_{\mathrm{N}} = \alpha\cos\theta\,\mathrm{e}^{-\frac{1}{2}\mathrm{i}\sigma t}J_1\left(\frac{a\nu r}{d}\right)\cosh\nu(z+1)+*+\text{decaying modes}$$
$$+B\cos 2\theta\,\mathrm{e}^{-\mathrm{i}\sigma t}J_2\left(\frac{a\mu r}{d}\right)\cosh\mu(z+1)+*+\text{decaying modes}, \tag{5.3}$$

where

$$k\tanh k = \frac{d\sigma^2}{g}, \quad \nu\tanh\nu = \frac{d\Omega_1^2}{g}, \quad \mu\tanh\mu = \frac{d\Omega_2^2}{g}. \tag{5.4}$$

Ω_1 represents the $\Omega_{\alpha mn}$ defined in §4 that lies closest to $\frac{1}{2}\sigma$ for the $\cos\theta$ mode, and Ω_2 is the $\Omega_{\beta mn}$ that is closest to σ for the $\cos 2\theta$ mode. The free-surface boundary conditions yield equations in the form of (4.3):

$$\alpha R_1 = \mathrm{i}\alpha^*(A_2 V_1 + B V_2), \tag{5.5}$$

$$B R_2 = \mathrm{i}\alpha^2 V_3, \tag{5.6}$$

where

$$R_1 = \left(\frac{d\Omega_1^2}{g}-\frac{d\sigma^2}{4g}\right)\cosh\nu\int_0^1 J_1^2\left(\frac{a\nu r}{d}\right)r\,\mathrm{d}r, \tag{5.7}$$

$$R_2 = \left(\frac{d\Omega_2^2}{g}-\frac{d\sigma^2}{g}\right)\cosh\mu\int_0^1 J_2^2\left(\frac{a\mu r}{d}\right)r\,\mathrm{d}r. \tag{5.8}$$

The functions V_1, V_2 and V_3 are defined in the appendix. The response for the linear forcing A_2 can be calculated from a method described in Yamamuro–Renardy (1981). A trivial solution is $\alpha = 0$ and $B = 0$. The questions to be resolved are whether there are any other solutions, and if so, under what conditions.

Eliminating B from (5.5) and (5.6) yields

$$\frac{\alpha}{\alpha^*} = \mathrm{i}\left(\frac{A_2 V_1}{V_2}+\mathrm{i}\alpha^2\frac{V_2 V_3}{R_1 R_2}\right). \tag{5.9}$$

Let

$$\alpha = |\alpha|\,\mathrm{e}^{\mathrm{i}\psi}, \quad f = \frac{-V_2 V_3}{R_1 R_2} = |f|\,\mathrm{e}^{\mathrm{i}\theta_1}, \quad g = \mathrm{i}\frac{A_2 V_2}{R_1} = |g|\mathrm{e}^{\mathrm{i}\theta_2}. \tag{5.10}$$

Then

$$|\alpha|^2 = \frac{[\cos\theta_1 \pm (\cos^2\theta_1 - 1 + |g|^2)^{\frac{1}{2}}]}{|f|}. \tag{5.11}$$

$$\psi = \tfrac{1}{2}\left(\theta_2 + \arcsin\frac{|\alpha|^2|f|\sin\theta_1}{|g|}\right). \tag{5.12}$$

Next, B is evaluated via

$$B = \mathrm{i}\alpha^2 V_3/R_2. \tag{5.13}$$

In order that there be non-trivial solutions, two conditions must be satisfied. First, $|\alpha|^2$ must be positive, and, from (5.11), the conditions that must be met are

(a) $\cos^2\theta_1 - 1 + |g|^2 \geqslant 0$,

(b) if $\cos\theta_1 + (\cos^2\theta_1 - 1 + |g|^2)^{\frac{1}{2}} > 0$ then there is at least one non-trivial $|\alpha|$. If

$$\cos\theta_1 - (\cos^2\theta_1 - 1 + |g|^2)^{\frac{1}{2}} > 0$$

then there are two solutions for $|\alpha|$. Secondly, (5.12) shows that $|\alpha|^2|f|/|g|$ must be less than or equal to 1.

If Ω_2 is close enough to σ, then k is approximately μ, so that the total velocity potential in the sill region is (approximately)

$$\phi \approx \alpha\cos\theta\,\mathrm{e}^{-\frac{1}{2}\mathrm{i}\sigma t} J_1\left(\frac{a\nu r}{d}\right)\cosh\nu(z+1) + (B+A_2)\cosh 2\theta\,\mathrm{e}^{-\mathrm{i}\sigma t} J_2\left(\frac{akr}{d}\right)\cosh k(z+1) + *. \tag{5.14}$$

This approximation may be used for the present example. In this case, computations revealed two ranges of forcing frequencies σ, in which near-resonance occurs. One range lies near $2\Omega_1$ and the other is near Ω_2. In most of these ranges, the wave amplitudes were calculated to be rather high, so that instabilities may occur, after which the present theory may not be applicable. However, at the upper end of the range near $2\Omega_1$, the amplitudes were found to be small enough so that the present steady-state theory might be observable in practice.

The author is indebted to Professor J. J. Mahony (University of Western Australia) for suggesting this topic and for many helpful discussions. Thanks are also due to Dr W. G. Pritchard (University of Essex) for help in drafting this paper. The work for this paper was supported by a Commonwealth Postgraduate Research Award, the National Science Foundation Grant no. MCS-7927062 and the U.S. Army Contract no. DAAG29-80-C-0041.

Appendix

$$\epsilon_m = \begin{cases} 1 & (m = 0), \\ 2 & (m \neq 0), \end{cases}$$

$$f(k) = \int_{-1}^{0} \cosh^2 k(z+1)\,\mathrm{d}x,$$

$$F_m = \frac{\epsilon_m \mathrm{i}^{-m} J_m\left(\dfrac{a\lambda}{D}\right)}{2\lambda\sinh\lambda}, \quad F_m^* = \frac{\epsilon_m \mathrm{i}^{-m} J'_m\left(\dfrac{a\lambda}{D}\right)}{2\lambda\sinh\lambda},$$

$$g(\lambda, k) = \int_{-1}^{0} \cosh\lambda(\delta z+1)\cosh k(z+1)\,\mathrm{d}z,$$

$$h(\lambda) = \int_{-1/\delta}^{0} \cosh^2\lambda(\delta z+1)\,\mathrm{d}z,$$

$$H_m\left(\frac{a\lambda}{D}\right) = H_m^{(1)}\left(\frac{a\lambda}{D}\right), \quad H'_m\left(\frac{a\lambda}{D}\right) = H_m^{(1)\prime}\left(\frac{a\lambda}{D}\right),$$

$$L(k,z) = J_m\left(\frac{ak}{d}\right)\cosh k(z+1) - \left[\frac{H_m^{(1)}\left(\frac{a\lambda}{D}\right)\cosh\lambda(\delta z+1)\,g(\lambda,k)}{H_m^{(1)\prime}\left(\frac{a\lambda_\pi}{D}\right)\lambda h(\lambda)} + \sum_{p=1}^{\infty}\frac{K_m\left(\frac{a\lambda_p}{D}\right)\cos\lambda_p(\delta z+1)\,g(\mathrm{i}\lambda_p,k)}{K_m'\left(\frac{a\lambda_p}{D}\right)\lambda_p h(\mathrm{i}\lambda_p)}\right]\frac{kJ_m'\left(\frac{ak}{d}\right)}{\delta},$$

$$L_n(k_n,z) = \cos k_n(z+1) - \left[\frac{H_m^{(1)}\left(\frac{a}{D}\right)\cosh\lambda(\delta z+1)\,g(\lambda,\mathrm{i}k_n)}{H_m^{(1)\prime}\left(\frac{a\lambda}{D}\right)\lambda h(\lambda)} + \sum_{p=1}^{\infty}\frac{K_m\left(\frac{a\lambda_p}{D}\right)\cos\lambda_p(\delta z+1)\,g(\mathrm{i}\lambda_p,\mathrm{i}k_n)}{K_m'\left(\frac{a\lambda_p}{D}\right)\lambda_p h(\mathrm{i}\lambda_p)}\right]\frac{k_n I_m'\left(\frac{ak_n}{d}\right)}{\delta I_m\left(\frac{ak_n}{d}\right)},$$

$$X(k,\lambda,\lambda_n) = J_m\left(\frac{ak}{d}\right)f(k) - \frac{kJ_m'\left(\frac{ak}{d}\right)}{\delta}\left[\frac{H_m\left(\frac{a\lambda}{D}\right)[g(\lambda,k)]^2}{H_m'\left(\frac{a\lambda}{D}\right)\lambda h(\lambda)} + \sum_{p=1}^{\infty}\frac{K_m\left(\frac{a\lambda_p}{D}\right)[g(\mathrm{i}\lambda_p,k)]^2}{K_m'\left(\frac{a\lambda_p}{D}\right)\lambda_p h(\mathrm{i}\lambda_p)}\right],$$

$$X_p(k_p,\lambda,\lambda_n) = \frac{-I_m'\left(\frac{ak_p}{d}\right)k_p}{I_m\left(\frac{ak_p}{d}\right)\delta}\left[\frac{H_m\left(\frac{a\lambda}{D}\right)g(\lambda,\mathrm{i}k_p)\,g(\lambda,\mathrm{i}k_p)}{H_m'\left(\frac{a\lambda}{D}\right)\lambda h(\lambda)} + \sum_{n=1}^{\infty}\frac{K_m\left(\frac{a\lambda_n}{D}\right)g(\mathrm{i}\lambda_n,k)\,g(\mathrm{i}\lambda_n,\mathrm{i}k_p)}{K_m'\left(\frac{a\lambda_n}{D}\right)\lambda_n h(\mathrm{i}\lambda_n)}\right],$$

$$Y(k,k_p,\lambda,\lambda_n) = -\frac{kJ_m'\left(\frac{ak}{d}\right)}{\delta}\left[\frac{H_m\left(\frac{a\lambda}{D}\right)g(\lambda,\mathrm{i}k_p)\,g(\lambda,k)}{H_m'\left(\frac{a\lambda}{D}\right)\lambda h(\lambda)} + \sum_{n=1}^{\infty}\frac{K_m\left(\frac{a\lambda_n}{D}\right)g(\mathrm{i}\lambda_n,\mathrm{i}k_p)\,g(\mathrm{i}\lambda_n,k)}{K_m'\left(\frac{a\lambda_n}{D}\right)\lambda_n h(\mathrm{i}\lambda_n)}\right],$$

$$Y(k_p,k_p,\lambda,\lambda_n) = f(\mathrm{i}k_p) - \frac{I_m'\left(\frac{ak_p}{d}\right)k_p}{I_m\left(\frac{ak_p}{d}\right)\delta}\left[\frac{H_m\left(\frac{a\lambda}{D}\right)[g(\lambda,\mathrm{i}k_p)]^2}{H_m'\left(\frac{a\lambda}{D}\right)\lambda h(\lambda)} + \sum_{n=1}^{\infty}\frac{K_m\left(\frac{a\lambda_n}{D}\right)[g(\mathrm{i}\lambda_n,\mathrm{i}k_p)]^2}{K_m'\left(\frac{a\lambda_n}{D}\right)\lambda_n h(\mathrm{i}\lambda_n)}\right],$$

$$Y(k_s, k_p, \lambda, \lambda_n)$$
$$= -\frac{k_s}{\delta}\frac{I'_m\left(\frac{ak_s}{d}\right)}{I_m\left(\frac{ak_s}{d}\right)}\left[\frac{H_m\left(\frac{a\lambda}{D}\right)g(\lambda, \mathrm{i}k_p)\,g(\lambda, \mathrm{i}k_s)}{H'_m\left(\frac{a\lambda}{D}\right)\lambda h(\lambda)} + \sum_{t=1}^{\infty}\frac{K_m\left(\frac{a\lambda_t}{D}\right)g(\mathrm{i}\lambda_t, \mathrm{i}k_p)\,g(\mathrm{i}\lambda_t, \mathrm{i}k_s)}{K'_m\left(\frac{a\lambda_t}{D}\right)\lambda_t h(\mathrm{i}\lambda_t)}\right],$$

$$x_2(k, k_n) = -\frac{kJ_n\left(\frac{ak}{d}\right)}{\delta}\sum_{p=1}^{\infty}\frac{K_m\left(\frac{a\lambda_p}{D}\right)[g(\mathrm{i}\lambda_p, k)]^2}{K'_m\left(\frac{a\lambda_p}{D}\right)\lambda_p h(\mathrm{i}\lambda_p)} - \sum_{p=1}^{\infty}\frac{A_{mp}}{A_m}\frac{I'_m\left(\frac{ak_p}{d}\right)}{I_m\left(\frac{ak_p}{d}\right)}\frac{k_p}{\delta}$$
$$\left[\frac{H_m\left(\frac{a\lambda}{D}\right)g(\lambda, k)\,g(\lambda, \mathrm{i}k_p)}{H'_m\left(\frac{a\lambda}{D}\right)\lambda h(\lambda)} + \sum_{n=1}^{\infty}\frac{g(\mathrm{i}\lambda_n, k)\,g(\mathrm{i}\lambda_n, \mathrm{i}k_p)\,K_m\left(\frac{a\lambda_n}{D}\right)}{\lambda_n h(\mathrm{i}\lambda_n)\,K'_m\left(\frac{a\lambda_n}{D}\right)}\right],$$

$$V_1 = -\tfrac{1}{2}[-2\,\overline{\nu\sinh\nu}(-\alpha^2 k\sinh k + k^2\cosh k) + k\sinh k(-\tfrac{1}{4}\alpha^2\,\overline{\nu\sinh\nu}$$
$$+\overline{\nu\sin\nu})]\int_0^1 J_1\left(\frac{a\nu r}{d}\right)J_2\left(\frac{akr}{d}\right)J_1\left(\frac{a\bar{\nu}r}{d}\right)r\,\mathrm{d}r$$
$$+\alpha^2\left(\tfrac{1}{2}k\cosh k\,\overline{\nu\cosh\nu}\int_0^1 J'_1\left(\frac{a\bar{\nu}r}{d}\right)J_1\left(\frac{a\nu r}{d}\right)J'_2\left(\frac{akr}{d}\right)r\,\mathrm{d}r\right.$$
$$+\frac{d^2}{a^2}\cosh k\,\overline{\cosh\nu}\int_0^1 J_1\left(\frac{a\bar{\nu}r}{d}\right)J_2\left(\frac{akr}{d}\right)J_1\left(\frac{a\nu r}{d}\right)\frac{\mathrm{d}r}{r}$$
$$\left.+\tfrac{1}{2}k\sinh k\,\overline{\nu\sinh\nu}\int_0^1 J_1\left(\frac{a\bar{\nu}r}{d}\right)J_1\left(\frac{a\nu r}{d}\right)J_2\left(\frac{akr}{d}\right)r\,\mathrm{d}r\right),$$

where $k\tanh k = d\sigma^2/g$.

V_2 is identical with V_1 but with μ instead of k, where $\mu\tanh\mu = d\Omega_2^2/g$.

$$V_3 = \nu^2\sinh\nu(-\nu\cosh\nu + \tfrac{3}{4}\alpha^2\sinh\nu)\int_0^1 J_2\left(\frac{a\mu r}{d}\right)J_1\left(\frac{a\bar{\nu}r}{d}\right)J_1\left(\frac{a\nu r}{d}\right)r\,\mathrm{d}r$$
$$+\frac{\alpha^2}{2}\left(\nu^2\cosh^2\nu\int_0^1 J_2\left(\frac{a\mu r}{d}\right)J'_1\left(\frac{a\bar{\nu}r}{d}\right)J'_1\left(\frac{a\nu r}{d}\right)r\,\mathrm{d}r\right.$$
$$\left.-\frac{d^2}{a^2}\cosh^2\nu\int_0^1 J_2\left(\frac{a\mu r}{d}\right)J_1\left(\frac{a\bar{\nu}r}{d}\right)J_1\left(\frac{a\nu r}{d}\right)\frac{\mathrm{d}r}{r}\right).$$

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